(CAPARIS ~~2012~~)

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MATHÉMATIQUES

On finite additive bases

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I Introduction

$B \subset \{0, 1, \dots, N-1\}$ is a 2-basis for N if $B + B \supseteq \{0, 1, \dots, N-1\}$

$$\binom{|B|}{2} + |B| \geq N \Rightarrow |B|^2 + |B| \geq 2N.$$

$$\sigma = \limsup_{N \to \infty} \frac{N}{|B|^2} \leq \frac{1}{2}$$

Best construction (Mrose 1979): $\sigma \geq \frac{2}{7} = 0,2857\dots$

Improvements for the upper bound:

0,4992	Rohrbach	1937
0,4903	Moser	1960
0,4867	Riddell	1960
0,4847	Moser, Pounder, Riddell	1969
0,4802	Klotz	1969
0,4789	Güntürk, Nathanson	2006
0,4775	Li An-Ping	2012
0,46972	Yu	2008

http://www.peerevaluation.org/read/libraryID:28270

Tools: Fourier expansions + (combinatorial lemmas)

Generating function: $f_B(t) = \sum_{b \in B} \exp(2i\pi b t)$

Representation function $r_B(n) = \#\{(a,b) \in B \times B : a+b = n\}$

Difference function $d_B(n) = \#\{(a,b) \in B \times B : a-b = n\}$

$$f_B(t)^2 = 2\sum_{n=0}^{N-1} \exp(2i\pi n t) + \sum_{n=0}^{2n-2} r^*(n) \exp(2i\pi n t)$$

avec $\begin{cases} r^*(n) \geq 0 & \text{si } n \notin 2B \\ r^*(n) \geq -1 & \text{si } n \in 2B \end{cases}$

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II A general approach

<u>Lemma</u>: For $|t| < 1/2$

$$\left| f_B(t)^2 - 2N\int_0^1 \exp(2i\pi N t_x)\, dt \right| \lesssim |B|^2 - 2N$$

Weight function: $w(x) = \sum_{k\in\mathbb{Z}} \hat{w}(k) \exp(2i\pi t_k x)$ with $t_0 = 0$

and $\|w\| = \sum_{k\in\mathbb{Z}} |\hat{w}(k)| < \infty$

Rk: If, $w_y(x) = w(x+y)$, $\hat{w}_y(k) = \exp(2i\pi t_k y)\, \hat{w}(k)$

so $\hat{w}(0)$ and $\|w\|$ are invariant.

$$R_B(w) = \sum_n w(n/N)\, r_B(n) = \sum_k \hat{w}(k)\, f_B(t_k/N)^2$$
$$D_B(w) = \sum_n w(n/N)\, d_B(n) = \sum_k \hat{w}(k)\, |f_B(t_k/N)|^2$$

<u>Lemma</u>:
$$\left| R_B(w) - 2N\int_0^1 w(x)\,dx \right| \lesssim \|w\| . (|B|^2 - 2N)$$

$$\min_{[0,2]} w \times (|B|^2 - 2N) \le R_B(w) - 2N\int_0^1 w(x)\,dx \le \max_{[0,2]} w \times (|B|^2 - 2N)$$

this does not give any improvement. We need something else:

- combinatorial lemma
- ~~[illegible]~~ $Nt \in \mathbb{Z} \setminus N\mathbb{Z} \Rightarrow |f_B(t)| \lesssim \sqrt{|B|^2 - 2N}$
- w 1-periodical $\Rightarrow \int_0^1 w(x)\,dx = \hat{w}(0)$
  and we ask for $\hat{w}(0) = 0$
- w p-periodical + relation to another 1-periodical function.

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III Using a combinatorial lemma

$$|f^2(\tfrac{n}{N})| \leq \sum_{n}^{n+N} |r^*(n)| = M^2 \qquad M \lesssim \sqrt{|B|^2 - 2N}$$

$$M^2 \geq \sum_{n \geq N} r(n) \geq \ell^2 \qquad \text{with } \ell = \# B \cap [N/2, N-1]$$

For $\varphi(t) = \sum_k \hat{\varphi}(k) \exp(2i\pi k t)$

$$S = \sum_{b \in B} \varphi\left(\frac{b}{N}\right) = \sum_k \hat{\varphi}(k)\, f_B\left(\frac{k}{N}\right)$$

$$(|B| - \ell) \min_{[0,1/2]} \varphi + \ell \min_{[1/2,1]} \varphi \leq S \leq \sum_{k \nmid N} |\hat{\varphi}(k)| \cdot M + \sum_{k \mid N} |\hat{\varphi}(k)| \cdot |B|$$

~~example 1~~: If $\min_{[1/2,1]} \varphi < \min_{[0,1/2]} \varphi$, we use $\ell \leq M$ to get a lower bound for M and an upper estimate for σ.

example 1: Moser $\varphi(t) = \sin(2\pi t) + \frac{1}{2}\cos(4\pi t) = \frac{1}{2} + \sin \pi t\,(1 - \sin \pi t)$

$$\min_{[0,1/2]} \varphi = \frac{1}{2} \qquad \min_{[1/2,1]} \varphi = -3/2$$

$$\sigma \leq \frac{1}{2}\left(1 - \frac{1}{49}\right) < 0{,}4898$$

example 2: Moser, Pounder, Riddell

$$\varphi(t) = \begin{cases} 1 & \text{if } t \in [0, 1/2] \\ 1 + \pi \sin(2\pi\theta) & \text{if } t \in [1/2, 1] \end{cases}$$

$$\sigma \leq \frac{1}{2}\left(1 - \left(1 + \frac{3\pi}{2}\right)^{-2}\right) \leq 0{,}4847$$

Extension 1: Güntürk, Nathanson

$$S = \sum_{b_1, b_2} \varphi\left(\frac{b_1}{N}, \frac{b_2}{N_2}\right) \qquad \varphi \text{ defined on } [0,1]^2$$

$$\varphi(t_1, t_2) = \begin{cases} 1 & \text{if } t_1 + t_2 < 1 \\ 1 - 40(1-t_1)(1-t_2)(1 - 2(1-t_1-t_2)^6) & \text{if } t_1 + t_2 \geq 1 \end{cases}$$

$\sigma \leq 0{,}4789$

<u>Extension 2</u> Li An Ping $\sigma \leq 0{,}4775$

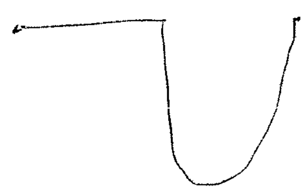
$$M^2 \geq \ell^2 + 2\ell_1 \ell_2 \quad \text{with} \quad \ell_1 = \#\mathcal{B} \cap [\theta N, N/2] \qquad \theta < 1/2$$
$$\ell_2 = \#\mathcal{B} \cap [(1-\theta) N, N]$$

$\theta = 3/8$

\+ 3 auxiliary functions to get inequalities s.t.

$$k - \frac{4\pi}{3}\ell_2 \leq 3{,}893\, M$$

obtained with
$$\varphi(x) = \begin{cases} 1 & 0 \leq x \leq 5/8 \\ 1 + \frac{4\pi}{3} \sin\left[\frac{8\pi}{3}(x-1)\right] & 5/8 \leq x \leq 1 \end{cases}$$



IV Using a period $p \geq 1$

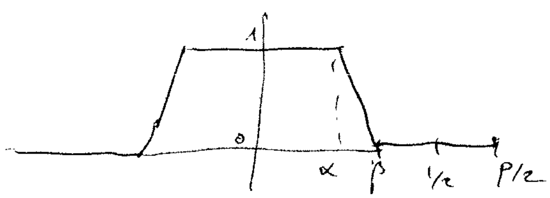
$$w(x) = \sum_k \hat{w}(k) \exp\left(\frac{2i\pi n x}{p}\right)$$

It is easier to estimate $\sum_k |\hat{w}(k)|$ if $\hat{w}(k) \geq 0$: then $= w(0)$.

Q: How to get $\hat{w}(k) \geq 0$ for all k?
A: $w = u * u$ with $\hat{u}(k) \in \mathbb{R}$.

Basic example: $0 \leq \alpha \leq \beta \leq \frac{1}{2} \leq p/2$

$$w_{p,\alpha,\beta}(x) = \begin{cases} 1 & 0 \leq |x| \leq \alpha \\ 1 - \frac{|\beta - |x||}{\beta - \alpha} & \alpha \leq |x| \leq \beta \\ 0 & \beta \leq |x| \leq p/2 \end{cases}$$



$$\begin{cases} \hat{w}_{p,\alpha,\beta}(0) = \frac{\alpha+\beta}{p} \\ \hat{w}_{p,\alpha,\beta}(n) = \frac{p}{2\pi^2 n^2} \times \frac{\cos\left(2\pi n \frac{\alpha}{p}\right) - \cos\left(2\pi n \frac{\beta}{p}\right)}{\beta - \alpha} \end{cases}$$

We get $\hat{w}_{p,0,\beta}(n) \geq 0$

Put $w_{p,\alpha,\beta,\kappa}(x) = w_{p,\alpha,\beta}(x+\kappa)$

$$R(w_{p,\alpha,\beta,\kappa}) \lesssim 2N \int_0^1 w_{p,\alpha,\beta,\kappa}(x)\,dx + 1 \times (|B|^2 - 2N)$$

$$= 2N \int_\kappa^{\kappa+1} w_{p,\alpha,\beta}(x)\,dx + |B|^2 - 2N = 2N I_\kappa + |B|^2 - 2N$$

$$R(w_{p,\alpha,\beta,\kappa}) \geq \frac{\alpha+\beta}{p}|B|^2 - \sum_{n \neq 0} |\hat{w}_{p,\alpha,\beta}(n)| \times \left|f_B\left(\frac{n}{pN}\right)\right|^2$$ independent of κ

We choose κ such that I_κ is minimal.

1^{st} study : $\alpha = 0$

For $|x| \le 1$, $\quad w_{p,0,\beta}(x) \le w_{1,0,\beta}(x)$

$$D(w_{p,0,\beta}) \le D(w_{1,0,\beta}) \underset{Yu}{\lesssim} |\beta|^2 - (1-\beta)2N$$

$$R(w_{p,0,\beta}) \ge \frac{2\beta|\beta|^2}{p} - D(w_{p,0,\beta})$$

$$\Rightarrow \quad \sigma \le \frac{1(1-\beta/p)}{1 - I_K - \beta} \qquad I_K = \frac{\left(\beta - \frac{p-1}{2}\right)^2}{p}$$

$\beta = 0{,}2837 \quad \Rightarrow \quad 0{,}4714$
$p = 1{,}3910$

2^{nd} study. $\quad \beta = 3\alpha$

(*) Yu : $w = w_{p_1,0,2\alpha,K_1} + w_{p_2,0,2\alpha,K_2}$ $\qquad 0{,}46972$

p_1, p_2 determined by the best K.

$p_1 = 1{,}275 \quad p_2 = 1{,}5889$
$K_1 = 0{,}1375 \quad K_2 = 0{,}2257$
$2\alpha = 0{,}2257$

(*) $\cos x - \cos 3x = 2\cos x\,(1 - \cos 2x)$ $\qquad 0{,}4757$

$\Rightarrow |\hat{w}_{p,\alpha,3\alpha}(n)| \le 2\,\hat{w}_{p,0,2\alpha}(n)$ $\qquad \alpha = 0{,}0894$, $p = 1{,}3991$

(*) $p = 2q\alpha \quad \beta = 3\alpha \qquad |\hat{w}_{p,\alpha,3\alpha}(n)| \le 2\cos\frac{\pi}{q}\,\hat{w}_{p,0,2\alpha}(n)$

$q = 6 \quad 0{,}47172 \quad \alpha = 0{,}1252814$
$q = 7 \quad 0{,}47192 \quad \alpha = 0{,}1037895$
$p = 13\alpha \quad \beta = 3 \quad 2\cos\frac{\pi}{13} \rightsquigarrow 0{,}47567 \quad \alpha = 0{,}11175..$

(*) $p = 12\alpha$, $\beta = 3\alpha \qquad x_n = \frac{\pi n}{6} \qquad |\cos x_n - \cos 3x_n| \le 1 - \cos 2x_n + \frac{\sqrt{3}-1}{2}(1 - \cos 3x_n)$

$\alpha = 0{,}1238662..$ $\qquad \Rightarrow 0{,}47072$